High dimensional sequential compactness

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High dimensional sequences

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If $M \in [\omega]^\omega$ and $f : [M]^n \to X$, we say that f converges to $x \in X$ if for every $x \in U \subseteq X$ open, there exists $k \in \omega$ such that $f''[M \setminus k]^n \subseteq U$.

n-sequentially compact spaces

Definition (Kubis & Szeptycki)

A space X is n-sequentially compact, if for every function $f : [\omega]^n \to X$ there is $M \in [\omega]^\omega$ such that $f \restriction [M]^n$ converges to some $x \in X$.

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- The case $n = 2$ was considered by M. Bojańczyk, E. Kopczyński and S. Toruńczyk, where they show that compact metric spaces are 2-sequentially compact and used this to prove that compact metric semigroups have idempotents naturally associated to the limits of a two dimensional sequence $f : [\omega]^2 \to X$.

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- for every $M \in [\omega]^\omega$ there exists $b \in \mathcal{B}$ an initial segment of M (i.e., $b \sqsubseteq M$).

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We identify $[\omega]^{<\omega}$ with the set of increasing finite sequences of natural numbers. Then let
$T(\mathcal{B}) = \{s \in [\omega]^{<\omega} : \exists b \in \mathcal{B}\ (s \subseteq b)\} \subseteq \omega^{<\omega}$.

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Examples of barriers

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- $\mathcal{S} = \{s \in [\omega]^{<\omega} : |s| = \text{mín}(s) + 1\}$ is a barrier of rank ω.

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Notice that $\mathcal{B}|N$ is a barrier on N and $\rho(\mathcal{B}|N) \leq \rho(\mathcal{B})$.

Infinite dimensional sequential compactness

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Let $\mathcal{B}$ be a barrier on $M \in [\omega]^\omega$. A $\mathcal{B}$-sequence in X is a function $f : \mathcal{B} \to X$. We say that f converges to $x \in X$ if for every $x \in U \subseteq X$ open, there exists $k \in \omega$ such that $f[\mathcal{B}|(M \setminus k)] \subseteq U$.

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Definition

A space is $\mathcal{B}$-sequentially compact (for a barrier $\mathcal{B}$ on ω), if for every $\mathcal{B}$-sequence, there exists $M \in [\omega]^\omega$ such that $f \restriction (\mathcal{B}|M)$ converges to some $x \in X$.

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Theorem (C., Guzmán, López-Callejas, Memarpanahi, Szeptycki, Todorčević)

The following are equivalent:

- X is α-sequentially compact,
- X is $\mathcal{B}$-sequentially compact for every "uniform" barrier of rank α,
- X is $\mathcal{B}$-sequentially compact for some "uniform" barrier $\mathcal{B}$ of rank α.

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Uniformity is a combinatorial property for a barrier $\mathcal{B}$, that ensures that the rank of $\mathcal{B}$ does not decrease if we take an infinite restriction $\mathcal{B}|M$.

Theorem

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If $\alpha < \beta < \omega_1$ and X is β-sequentially compact, then X is also α-sequentially compact.

The case $\alpha, \beta \in \omega$ was previously proved by Kubis and Szeptycki.

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Theorem

- (Blass) $\mathfrak{par}_1 = \mathfrak{s}$,
- (Blass) $\mathfrak{par}_n = \mathfrak{par}_2$ for every $1 < n \in \omega$,
- (CGLMST) $\mathfrak{par}_{\mathcal{B}} = \mathfrak{par}_2$ for every barrier $\mathcal{B}$,
- (Kubis & Szeptycki) $\mathfrak{par}_2$ is the minimum κ such that 2^κ is not n-sequentially compact,
- (CGLMST) $\mathfrak{par}_2$ is the minimum κ such that 2^κ is not $\mathcal{B}$-sequentially compact for every barrier $\mathcal{B}$.

ω_1-sequentially compact spaces.

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Theorem (C., Guzmán and López-Callejas)

The cardinal invariant $\mathfrak{b}$ is characterized as the minimum character of a sequentially compact space that is not 2-sequentially compact.

Theorem (Todorčević)

Every compact bisequential space is ω_1-sequentially compact.

Recall that X is bisequential at x if every ultrafilter converging to x contains a countable family converging to x too. The space X is bisequential if it is bisequential at every point.

sequentially compact + $\chi(X) < \mathfrak{b}$ $\searrow$ $\swarrow$ Compact bisequential

$\omega_1 -$ sequentially compact

$\downarrow$

$\vdots$

$\downarrow$

$\alpha -$ sequentially compact

$\downarrow$

$\vdots$

$\downarrow$

$2 -$ sequentially compact

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Theorem (C., Guzmán, López-Callejas)

There is a Fréchet sequentially compact space that is not 2-sequentially compact. For every $n \in \omega$, there is an n-sequentially compact space that is not $(n+1)$-sequentially compact if one assumes any of the following:

- $\mathfrak{b} = \mathfrak{c}$
- $\diamondsuit(\mathfrak{b}) + \mathfrak{d} = \omega_1$
- $\mathfrak{s} = \mathfrak{b}$

Theorem(CGLMST)

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A more general result can be proved: There are two natural ideals associated to a barrier $\mathcal{B}$, namely $\mathrm{Fin}^{\mathcal{B}}$ and $\mathcal{G}_c(\mathcal{B})$. Then if $\mathcal{B}$ and $\mathcal{C}$ are two barriers and $\mathrm{FIN}^{\mathcal{C}} \not\leq_K \mathcal{G}_c(\mathcal{B})$, there is a $\mathcal{B}$-sequentially compact space that is not $\mathcal{C}$-sequentially compact.

Some classes spaces that are ω_1-sequentially compact

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- Rosenthal compact spaces.

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Corollary

If a space in any of the previous classes has a semigroup structure with continuous multiplication, then it has a "nice" idempotent.

Angelic spaces and the Ramsey property

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Definition (H. Knaust)

A space X has the Ramsey property if for every double sequence $\{x_{n,m} : n < m < \omega\}$ such that $\text{lím}_{n\to\infty} \text{lím}_{m\to\infty} x_{n,m} = x$, the function $f : [\omega]^2 \to X$ given by $f(\{n, m\}) = x_{n,m}$ also converges to x.

Angelic spaces with the Ramsey property

Theorem (Knaust)

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Theorem (CGLMST)

There is (in ZFC) an angelic space without the Ramsey property.

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- Applications to topological groups.
- Applications to Banach spaces
- (Currently working on) High dimensional versions of other kind of compactness and convergence.

References

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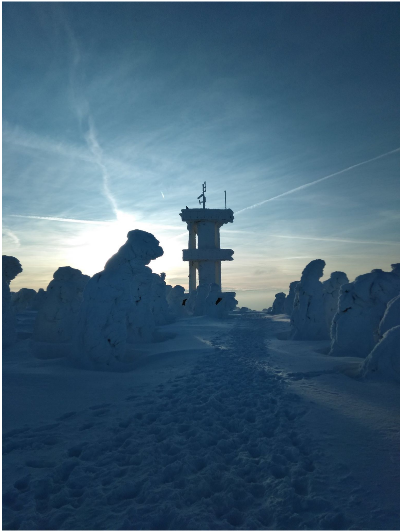

Thank you!